INT. J. NUM. METH. HEAT FLUID FLOW, VOL. 3, 49–62 (1993)

USE OF ISOTROPIC FUNDAMENTAL SOLUTIONS FOR HEAT CONDUCTION IN ANISOTROPIC MEDIA

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ABSTRACT

A numerical formulation for solving homogeneous anisotropic heat conduction problems based on the use of an isotropic fundamental solution is presented in detail. The analysis is carried out assuming a generic position of the coordinate axes, which may not coincide with the principal directions of orthotropy of the material. The two primary integral equations of the method are derived from the governing differential equation of the problem. Then, the numerical procedure is developed by rewriting the internal degrees of freedom that arise from the domain discretization in terms of the boundary nodes and solving the resulting system of linear equations for the boundary unknowns only. Special attention is given to the differentiation of singular integrals which yields additional terms as well as to the evaluation of the resulting Cauchy principal value integral. The main feature of the proposed formulation is its generality, which makes possible its direct extension to solve the problem of three-dimensional heat conduction in anisotropic media and, foremost, to three-dimensional orthotropic and anisotropic elasticity or elastoplasticity.

KEY WORDS Anisotropic heat conduction Boundary element method

INTRODUCTION

The increasing number of industrial applications of anisotropic materials has attracted the attention of many researchers concerned with computational modelling. To solve current technological problems that occur, for instance, in the aerospace industry, the use of metals that have undergone heavy cold pressing, fibre-reinforced structures and heat shielding materials for extremely high temperatures[1] is sometimes essential. In fact many applications of fibre-reinforced laminates, which are regarded as the most common anisotropic material[2], are currently seen as conventional practice in engineering design. One clear example is the use of single and multilayer fibre-glass integrated circuit boards, which can be considered as thermal anisotropic plates[3].

The treatment of the heat conduction equation for anisotropic materials is generally regarded as difficult[1,4], especially for finite regions[1]. Therefore, to provide answers to realistic industrial problems, it is necessary to resort to numerical methods.

In this context, the boundary element method (BEM)[5–7] is acquiring considerable popularity among engineers as it allows, in many cases, an impressive saving in the amount of time necessary to discretize the model when compared to the finite difference method and the finite element method (FEM). The numerical solution of problems of heat conduction in anisotropic media using Green's functions was first reported by Chang *et al.*[4].

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0961–5539/93/010049–14$2.00

Received May 1992
Revised July 1992

This paper is concerned with the development of an alternative integral equation formulation which makes use of an isotropic fundamental solution for solving heat conduction problems in homogeneous anisotropic media. The main feature of this approach is its generality, which enables a straightforward and systematic extension to solve two- and three-dimensional anisotropic elastic problems. In this sense, although the two-dimensional anisotropic fundamental solution presents no particular difficulty in its implementation[8], the evaluation of the contour integrals for the three-dimensional case is quite complex, especially for the fundamental tractions[9], and would be too time consuming for routine numerical use[10]. Another approach to the problem was presented by Kinoshita and Mura[11] where convergent series expansions for the fundamental solution and its derivatives are discussed, though, according to Wilson and Cruse[10], this formulation is not suitable for extensive computation either. These difficulties make the extension aforementioned attractive, in particular to improve computational efficiency.

The idea of using an isotropic fundamental solution to solve anisotropic problems was first reported by Niwa *et al.*[12], in an integral formulation to solve three-dimensional non-homogeneous anisotropic elastodynamic problems. Nevertheless, in this formulation, the homogeneous isotropic fundamental solution was used because an anisotropic elastodynamic one was not found in the literature. Later, Brebbia and Dominguez[7] suggested the use of the isotropic elastic fundamental solution in an iterative formulation for anisotropic elastostatics, albeit no results were presented to validate the proposed method. Recently, Shi[13] presented an integral formulation for anisotropic materials in the context of square orthotropic plates.

The formulation proposed here draws its inspiration from the work of Shi[13], along with the previous works of Mikhlin[14,15] and Telles and Brebbia[16] in computing derivatives of strongly singular integrals. Moreover, it is an extension of the method proposed by Perez and Wrobel[17] for orthotropic problems in potential theory.

The present approach consists of rewriting the governing differential equation of the problem in a slightly different form to enable the application of the direct BEM formulation with the isotropic fundamental solution. This procedure leads to the first primary integral equation of the method.

To cope with the resulting domain unknowns, a supplementary integral equation is derived from the original one. In its derivation, special care is taken in treating domain integrals that involve strongly singular kernels.

To solve a problem by using the formulation proposed here it is necessary to discretize the contour into boundary elements and the domain into internal cells. In the present paper the use of constant boundary elements along with constant internal cells is adopted for the sake of simplicity. This discretization is applied to the primary integral equations of the method to obtain the related system of linear matrix equations. These matrix equations are then treated by using a technique equivalent to the FEM condensation of internal degrees of freedom[18,19], leading to a final solution that is dependent exclusively on the boundary variables of the problem. Although details of the mathematical development are only given for two spatial dimensions, the basic idea behind the present method is not limited to plane problems.

To assess the accuracy of the proposed formulation, two illustrative problems are considered in this work. In the first example analysed, the results obtained by the present method are compared with an analytical solution to the problem and with results obtained by a transformation of coordinates which enables the use of a conventional BEM program to solve the resulting Laplace's equation in the transformed space. The second example verifies the influence of the internal cells on the convergence of the method, by analysing an anisotropic strip using three different internal meshes. The results obtained for these discretizations are compared with the analytical solution to the problem, presented by Xiangzhou[20].

Further developments to the method proposed here are suggested at the end of this work, where applications to elasticity and elastoplasticity are briefly considered.

INTEGRAL EQUATION FORMULATION

The governing differential equation for two-dimensional steady-state heat conduction in an anisotropic medium in the absence of internal heat generation is of the form:

$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial x}\left(k_{xy}\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(k_{yx}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{yy}\frac{\partial T}{\partial y}\right)=0 \qquad (1)$$

where T is the temperature; and k_{xx}, k_{xy}, k_{yx}, and k_{yy} are the coefficients of thermal conductivity of the medium. This first equation embodies the concept of general anisotropy, which refers to a non-homogeneous medium.

In this paper, only homogeneous materials are considered. In addition, from Onsager's theorem of the thermodynamics of irreversible processes[1,21], it is shown that when the fluxes and the temperature gradients are related to each other linearly, as implied in (1), the conductivity coefficients obey the reciprocity relation:

$$k_{xy}=k_{yx} \qquad (2)$$

Thus, taking into account these considerations, (1) can be rewritten as:

$$k_{xx}\frac{\partial^2 T}{\partial x^2}+2k_{xy}\frac{\partial^2 T}{\partial x\partial y}+k_{yy}\frac{\partial^2 T}{\partial y^2}=0 \qquad (3)$$

It is also worth recalling that, as a consequence of the linear relation implied in (1), the heat flux is not necessarily normal to the isotherm passing through the point considered[1].

For convenience of the mathematical formulation, (3) is rewritten in the form:

$$\nabla^2 T=k_1\frac{\partial^2 T}{\partial y^2}-k_2\frac{\partial^2 T}{\partial x\partial y} \qquad (4)$$

where k_1 expresses the ratio $(k_{xx}-k_{yy})/k_{xx}$ and k_2 expresses the ratio $2k_{xy}/k_{xx}$.

Through the application of either Green's second identity or a weighted residual technique in which the right-hand side of the previous equation is considered as a fictitious heat source term, (4) is transformed into the following equivalent integral equation:

$$T(\xi)+\int_\Gamma T(\chi)f_n^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi)-\int_\Gamma f_n(\chi)T^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi)=$$

$$\int_\Omega\left[k_1\frac{\partial^2 T(\chi)}{\partial y^2}-k_2\frac{\partial^2 T(\chi)}{\partial x\partial y}\right]T^*(\xi,\chi)\,\mathrm{d}\Omega(\chi) \qquad (5)$$

where ξ and χ are the source and field points, respectively; $T^*(\xi,\chi)$ is the fundamental solution of the two-dimensional Laplace's equation (i.e. $T^*(\xi,\chi)=(1/2\pi)\ln(1/r(\xi,\chi))$, with $r(\xi,\chi)$ denoting the distance between ξ and χ); $f_n(\chi)=\partial T(\chi)/\partial n(\chi)$ is a force[1] at point χ, where n is the outward normal vector; $f_n^*(\xi,\chi)=\partial T^*(\xi,\chi)/\partial n(\chi)$ is the derivative of the fundamental solution T^* with respect to n; Ω is the domain of the anisotropic medium and Γ its boundary.

An integral equation similar to (5) can be obtained for boundary points ξ by a limiting process, leading to:

$$c(\xi)T(\xi) + \int_\Gamma T(\chi)f_n^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi) - \int_\Gamma f_n(\chi)T^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi) = \int_\Omega [k_1 b_1(T) - k_2 b_2(T)]\,T^*(\xi,\chi)\,\mathrm{d}\Omega(\chi) \tag{6}$$

where $c(\xi)$ is a function of the internal angle of the boundary at point ξ; $b_1(T) = \partial^2 T(\chi)/\partial y^2$ and $b_2(T) = \partial^2 T(\chi)/\partial x \partial y$. Then, the term between the square brackets in (6) is regarded as an unknown domain variable which should be computed along with the unknown boundary variables. This assumption, although not essential, leads to a significant reduction in the number of matrix operations necessary to determine the boundary solution, when compared with considering each term within the square brackets as a separate domain variable.

Hence, the use of the isotropic fundamental solution to solve this problem implies that another integral equation must be established in order to cope with the internal variables. This supplementary equation is obtained by means of a linear combination of two second-order derivatives of (5). The first one is the second derivative of (5) with respect to the coordinate y of the source point ξ, y_ξ, as follows:

$$\frac{\partial^2 T(\xi)}{\partial y_\xi^2} + \frac{\partial^2}{\partial y_\xi^2}\int_\Gamma T(\chi)f_n^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi) - \frac{\partial^2}{\partial y_\xi^2}\int_\Gamma f_n(\chi)T^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi) = \frac{\partial^2}{\partial y_\xi^2}\int_\Omega [k_1 b_1(T) - k_2 b_2(T)]\,T^*(\xi,\chi)\,\mathrm{d}\Omega(\chi) \tag{7}$$

The second equation is obtained from cross-differentiation of (5) with respect to x_ξ and y_ξ, which leads to:

$$\frac{\partial^2 T(\xi)}{\partial x_\xi \partial y_\xi} + \frac{\partial^2}{\partial x_\xi \partial y_\xi}\int_\Gamma T(\chi)f_n^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi) - \frac{\partial^2}{\partial x_\xi \partial y_\xi}\int_\Gamma f_n(\chi)T^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi) = \frac{\partial^2}{\partial x_\xi \partial y_\xi}\int_\Omega [k_1 b_1(T) - k_2 b_2(T)]\,T^*(\xi,\chi)\,\mathrm{d}\Omega(\chi) \tag{8}$$

Finally, by multiplying (7) by k_1 and (8) by k_2, and subtracting the latter from the former, the sought supplementary integral equation is established as:

$$\begin{aligned}&\{k_1 b_1[T(\xi)] - k_2 b_2[T(\xi)]\} + \frac{\partial^2}{\partial y_\xi^2}\int_\Gamma k_1 T(\chi)f_n^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi) - \\ &\frac{\partial^2}{\partial x_\xi \partial y_\xi}\int_\Gamma k_2 T(\chi)f_n^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi) - \\ &\frac{\partial^2}{\partial y_\xi^2}\int_\Gamma k_1 f_n(\chi)T^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi) + \frac{\partial^2}{\partial x_\xi \partial y_\xi}\int_\Gamma k_2 f_n(\chi)T^*(\xi,\chi)\,\mathrm{d}\Gamma(\chi) = \\ &\frac{\partial^2}{\partial y_\xi^2}\int_\Omega [k_1 b_1(T) - k_2 b_2(T)]k_1 T^*(\xi,\chi)\,\mathrm{d}\Omega(\chi) - \\ &\frac{\partial^2}{\partial x_\xi \partial y_\xi}\int_\Omega [k_1 b_1(T) - k_2 b_2(T)]k_2 T^*(\xi,\chi)\,\mathrm{d}\Omega(\chi)\end{aligned} \tag{9}$$

where the common term between square brackets in the kernel of both domain integrals turns out to be the same as the one between the curly brackets at the beginning of the equation. Equations (6) and (9), together with the boundary conditions, are then sufficient to determine the unknown variables of the problem.

Differentiation of the singular integrals

There is no difficulty in the differentiation of the boundary integrals in (9), as they are regular integrals. On the other hand, the treatment of the domain terms in (9) deserves special attention as the concept of differentiation of strongly singular integrals does not follow the classical rule[14,15]. The differentiation of these latter integrals leads in fact to additional terms, which can be determined through the use of Leibnitz' rule* [6,16,22].

Initially, following the treatment used by Telles and Brebbia[16], the expression for the derivative of the first singular domain integral in (9) should be written in a more formal representation, assuming the form:

$$\frac{\partial^2}{\partial y_\xi^2}\int_\Omega [k_1 b_1(T) - k_2 b_2(T)] k_1 T^*(\xi,\chi)\,d\Omega(\chi) =$$
$$k_1 \lim_{\varepsilon\to 0}\left\{\frac{\partial^2}{\partial y_\xi^2}\int_{\Omega_\varepsilon}[k_1 b_1(T) - k_2 b_2(T)]\, T^*(\xi,\chi)\,d\Omega(\chi)\right\} \tag{10}$$

where Ω_ε arises from Ω by removing a circle of radius ε centred at the source point ξ. This means that this integral is to be interpreted in the Cauchy principal value sense. For simplicity and without loss of generality, as the integration in (10) will be evaluated over each constant internal cell, the term $[k_1 b_1(T) - k_2 b_2(T)]$ can be considered as an unknown constant that will multiply the result obtained for each internal cell.

In order to deal with the singularity, it is possible to define a polar coordinate system $(\bar{r}, \bar{\theta})$ based at the point $O \equiv \xi$, as shown in *Figure 1a*. Then, if a small increment in the rectangular coordinate y of the singular point is given, not only r and ϕ become different from $\bar{r}$ and $\bar{\theta}$ but also Γ_ε is shifted, as seen in *Figure 1b*, indicating their dependence on the coordinates of the source point.

Applying the mentioned transformation of coordinates to (10), the following expression arises:

$$I_1 = k_1 \int_0^{2\pi} \lim_{\varepsilon\to 0}\left\{\frac{\partial^2}{\partial y_\xi^2}\int_{\bar{\varepsilon}(\bar{\theta})}^{R(\bar{\theta})} T^* \bar{r}\,d\bar{r}\right\} d\bar{\theta} \tag{11}$$

Taking one derivative at a time of the integral between the curly brackets it is possible to apply Leibnitz' rule twice, first obtaining:

$$\frac{\partial}{\partial y_\xi}\int_{\bar{\varepsilon}(\bar{\theta})}^{R(\bar{\theta})} T^*\bar{r}\,d\bar{r} = \int_{\bar{\varepsilon}(\bar{\theta})}^{R(\bar{\theta})} \frac{\partial T^*}{\partial y_\xi}\bar{r}\,d\bar{r} - T^*(\varepsilon)\bar{\varepsilon}\frac{\partial \bar{\varepsilon}}{\partial y_\xi} + T^*(R)R\frac{\partial R}{\partial y_\xi} \tag{12}$$

where the last term is equal to zero, as the vector R is independent of the position of the source point (see *Figure 1*).

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* According to Leibnitz' rule:

$$\frac{d}{d\alpha}\int_{\rho_1(\alpha)}^{\rho_2(\alpha)} F(x,\alpha)\,dx = \int_{\rho_1(\alpha)}^{\rho_2(\alpha)} \frac{\partial F}{\partial \alpha}\,dx - F(\rho_1,\alpha)\frac{d\rho_1}{d\alpha} + F(\rho_2,\alpha)\frac{d\rho_2}{d\alpha}$$

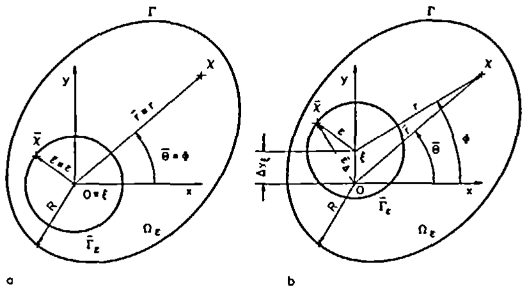

Figure 1 (a) Polar coordinate system based at $O \equiv \xi$, and (b) effect of an increment Δy_ξ in the rectangular coordinate y

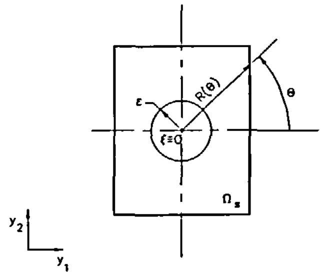

Figure 2 Nomenclature for the analytical evaluation of the singular integrals

Next, taking into consideration that $\xi \equiv O$ and that this leads to the distance $\bar{\varepsilon}$ being equal to the radius ε (*Figure 1a*), which does not depend on the position of the source point, it is possible to take the limit of this expression at this point, thus obtaining:

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial y_\xi} \int_{\bar{\varepsilon}(\bar{\theta})}^{R(\bar{\theta})} T^* \bar{r}\, \mathrm{d}\bar{r} = \fint_0^{R(\bar{\theta})} \frac{\partial T^*}{\partial y_\xi} \bar{r}\, \mathrm{d}\bar{r} \tag{13}$$

as the second term on the right hand side of (12) vanishes due to the weak singularity presented by the product $T^*(\varepsilon)\bar{\varepsilon}$ when y_ξ and ε tend to zero simultaneously. The second integral in (13) must be interpreted in the CPV sense, which is henceforth denoted by the dash through the integration symbol.

Finally, the second derivative of the integral between the curly brackets in (11) can be obtained by applying Leibnitz' rule to the resulting integral in (13), which gives:

$$\frac{\partial}{\partial y_\xi} \fint_0^{R(\bar{\theta})} \frac{\partial T^*}{\partial y_\xi} \bar{r}\, \mathrm{d}\bar{r} = \fint_0^{R(\bar{\theta})} \frac{\partial^2 T^*}{\partial y_\xi^2} \bar{r}\, \mathrm{d}\bar{r} - \frac{\partial T^*(\varepsilon)}{\partial y_\xi} \bar{\varepsilon} \frac{\partial \bar{\varepsilon}}{\partial y_\xi} + \frac{\partial T^*(R)}{\partial y_\xi} R \frac{\partial R}{\partial y_\xi} \tag{14}$$

where, analogously to what happened in (12), the last term vanishes as R does not depend on y_ξ.

In the second term on the right-hand side of (14) it is worth looking in greater detail at the derivative of $\bar{\varepsilon}$ with respect to y_ξ but first it is important to stress that $\bar{\varepsilon}$ is the distance from the origin of the polar system of coordinates to a field point on $\bar{\Gamma}_\varepsilon$, denoted $\bar{\chi}$ and not from the source point to $\bar{\chi}$, although both distances initially coincide (compare *Figure 1a* and *1b*). Then, it should be recalled that as the source point ξ is moved the circle $\bar{\Gamma}_\varepsilon$ goes along with it, making it necessary to apply the formal definition to differentiate $\bar{\varepsilon}$ with respect to y_ξ, as expressed by:

$$\frac{\partial \bar{\varepsilon}}{\partial y_\xi} = \lim_{\Delta y_\xi \to 0} \frac{\bar{\varepsilon}_\Delta - \bar{\varepsilon}}{\Delta y_\xi} \tag{15}$$

where $\bar{\varepsilon}$ is the distance from the origin of the system of coordinates to a point defined on the boundary $\bar{\Gamma}_\varepsilon$, *Figure 1a*; Δy_ξ is the displacement applied to the position of the source point in the direction y in order to evaluate the derivative; and $\bar{\varepsilon}_\Delta$ is the distance from the origin of the system of coordinates to the same reference point on the boundary $\bar{\Gamma}_\varepsilon$ that is now displaced Δy_ξ from its original position (*Figure 1b*).

The derivative $\partial/\partial y_\xi$, (15), is obtained by first expanding $\bar{\varepsilon}_\Delta$ according to the cosine theorem applied to the triangle $O\xi\bar{\chi}$ and then taking the limit as Δy_ξ tends to zero. This enables to determine $\partial\bar{\varepsilon}/\partial y_\xi$ as being equal to $-\sin\theta$ and, consequently, the second term in the right-hand side of (14) can now be written as:

$$\frac{\partial u^*(\varepsilon)}{\partial y_\xi}\bar{\varepsilon}\frac{\partial \bar{\varepsilon}}{\partial y_\xi} = \frac{1}{2\pi}\sin^2\theta \tag{16}$$

The final expression for the second derivative of the domain integral with respect to the coordinate y of the source point (11) is obtained by taking the limit of (14) as ε tends to zero and then transforming the resulting expression back to the rectangular coordinate system, leading to:

$$I_1 = k_1\left[\oint_\Omega \frac{\partial^2 T^*(\xi,\chi)}{\partial y_\xi^2}\mathrm{d}\Omega(\chi) - \frac{1}{2}\right] \tag{17}$$

where the integral can be evaluated semi-analytically for the type of internal discretization adopted in this work and the second term on the right-hand side can be seen either as resulting from the application of Leibnitz' rule or, as expressed by Bui[22], as the convected term due to the fact that the domain Ω_ε changes with the position of the source point ξ.

The same treatment was applied to the second domain integral in (9), leading to:

$$I_2 = k_2\left[\oint_\Omega \frac{\partial^2 T^*(\xi,\chi)}{\partial y_\xi \partial x_\xi}\mathrm{d}\Omega(\chi) - 0\right] \tag{18}$$

Accordingly, (9) can be rewritten in a more compact form as follows:

$$\{k_1 b_1[T(\xi)] - k_2 b_2[T(\xi)]\} + \int_\Gamma T(\chi)\left[k_1\frac{\partial^2 f_n^*(\xi,\chi)}{\partial y_\xi^2} - k_2\frac{\partial^2 f_n^*(\xi,\chi)}{\partial x_\xi \partial y_\xi}\right]\mathrm{d}\Gamma(\chi) -$$
$$\int_\Gamma f_n(\chi)\left[k_1\frac{\partial^2 T^*(\xi,\chi)}{\partial y_\xi^2} - k_2\frac{\partial^2 T^*(\xi,\chi)}{\partial x_\xi \partial y_\xi}\right]\mathrm{d}\Gamma(\chi) =$$
$$\sum_{i=1}^{m}(\{k_1 b_1[T(\chi_i)] - k_2 b_2[T(\chi_i)]\}[I_{1_i} - I_{2_i}]) \tag{19}$$

where m is the number of internal cells obtained in the discretization procedure and χ_i is the collocation point at the centre of internal cell i.

For the sake of the numerical formulation it is worth mentioning that the term between braces at the beginning of (19) will eventually be added to the similar one under the summation symbol after the collocation technique is applied to the left-hand side of the same equation. This procedure is especially convenient to avoid the need to compute $b_1[T(\xi)]$ and $b_2[T(\xi)]$ numerically, which would involve a larger number of matrix operations.

MATRIX FORMULATION

The numerical solution of the system of equations established by (6) and (19) is obtained by dividing both the contour Γ into l constant boundary elements and the domain Ω into a mesh of m rectangular cells, within which the unknown domain variable is regarded as constant. In the internal discretization collocation points are defined at the centroid of each cell. Then, applying the discrete version of (6) at each boundary node, the first set of linear equations is obtained in the form:

$$\mathbf{HT} - \mathbf{GF} = \mathbf{EB} \tag{20}$$

where **H** and **G** are the conventional $l \times l$ BEM influence matrices; $\mathbf{T} = T_i$ $(i = 1,\ldots, l)$ and $\mathbf{F} = f_{n_i}$ $(i = 1,\ldots, l)$ are the boundary temperature and normal boundary force vectors, respectively; **E** is an $l \times m$ matrix resulting from the domain integration; and $\mathbf{B} = b_i\,(i = 1,\ldots, m)$ is the vector of domain unknowns $[k_1 b_1(T) - k_2 b_2(T)]$ at the internal collocation points.

Another set of equations can analogously be obtained by applying the discretized version of (19) at the m internal collocation points, in the form:

$$\bar{\mathbf{H}}\mathbf{T} - \bar{\mathbf{G}}\mathbf{F} = \bar{\mathbf{E}}\mathbf{B} \tag{21}$$

where $\bar{\mathbf{H}}$ and $\bar{\mathbf{G}}$ are $m \times l$ matrices concerned with the boundary integrals whilst $\bar{\mathbf{E}}$ is an $m \times m$ matrix resulting from the domain integral.

Equations (20) and (21) contain $l + m$ variables, that is, the conventional l boundary unknowns plus the m unknown domain terms. Eliminating the m domain unknowns in vector **B**, (21) can be written in the following form:

$$\mathbf{B} = \bar{\mathbf{E}}^{-1}(\bar{\mathbf{H}}\mathbf{T} - \bar{\mathbf{G}}\mathbf{F}) \tag{22}$$

where $\bar{\mathbf{E}}^{-1}$ is the inverse of matrix $\bar{\mathbf{E}}$.

The expression for vector **B** in (22) can be substituted into (20), reducing it to:

$$(\mathbf{H} - \mathbf{E}\bar{\mathbf{E}}^{-1}\bar{\mathbf{H}})\mathbf{T} = (\mathbf{G} - \mathbf{E}\bar{\mathbf{E}}^{-1}\bar{\mathbf{G}})\mathbf{F} \tag{23}$$

This procedure is equivalent to the FEM condensation of internal degrees of freedom[18,19].

The final system of linear equations can be obtained by substituting the prescribed boundary data and rearranging (23) in order to obtain an expression of the form:

$$\mathbf{A}\mathbf{X} = \mathbf{P} \tag{24}$$

from where the boundary unknowns **X** of the problem are computed. Once this solution is obtained, the domain unknowns can be computed by referring to (5).

Evaluation of the Cauchy principal value integrals

Due to the type of internal discretization adopted in this work it is possible to evaluate the components of matrix $\bar{\mathbf{E}}$ using a semi-analytical approach. The integrals over the cell that contains the source point ξ were determined analytically while the integration over the non-singular cells was computed numerically using standard Gaussian quadrature.

In order to integrate analytically over the cell that contains the source point, Ω_s, the domain integral in (17) is expanded as:

$$I_{1_s} = \int_{\Omega_s} -\frac{1}{2\pi r^2}\left\{1 - \frac{2[y(\chi) - y(\xi)]}{r^2}\right\} \mathrm{d}\Omega(\chi) \tag{25}$$

Next, a polar system of coordinates centred at the source point is introduced, enabling to rewrite (25) in the form:

$$I_{1_s} = \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_{\varepsilon}^{R(\theta)} -\frac{1}{2\pi r}(1 - 2\sin^2\theta)\,\mathrm{d}r\,\mathrm{d}\theta \tag{26}$$

where $R(\theta)$ is the distance from ξ to the cell contour (see *Figure 2*).

The right-hand side of (26) can then be integrated with respect to r, leading to:

$$I_{1_s} = -\frac{1}{2\pi}\int_0^{2\pi}(1 - 2\sin^2\theta)\ln R(\theta)\,\mathrm{d}\theta - \left(-\frac{1}{2\pi}\right)\lim_{\varepsilon\to 0}\left(\ln\varepsilon\int_0^{2\pi}(1 - 2\sin^2\theta)\,\mathrm{d}\theta\right) \tag{27}$$

As the origin of the polar system of coordinates O, *Figure 2*, coincides with the source point ξ at the centre of the internal cell, the last integral in (27) is identically zero. In addition to that, the symmetry of the cell and of $R(\theta)$ makes the first integral in (27) also equal to zero, thus making I_{1_s} identically equal to zero.

Following similar approach, (18) can be expanded as:

$$I_{2_s} = \int_{\Omega_s} -\frac{1}{2\pi r^2}\left\{\frac{[y(\chi)-y(\xi)][x(\chi)-x(\xi)]}{r^2}\right\} d\Omega(\chi) \tag{28}$$

which, after the introduction of the polar system of coordinates centred at ξ can be rewritten in the form:

$$I_{2_s} = \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{R(\theta)} -\frac{1}{2\pi r}\sin\theta\cos\theta \, dr \, d\theta \tag{29}$$

The right-hand side of (29) was then integrated with respect to r, similarly to what was done in (26), thus leading to:

$$I_{2_s} = -\frac{1}{2\pi}\int_0^{2\pi}\sin\theta\cos\theta\ln R(\theta)\, d\theta - \left(-\frac{1}{2\pi}\right)\lim_{\varepsilon\to 0}\left(\ln\varepsilon\int_0^{2\pi}\sin\theta\cos\theta\, d\theta\right) \tag{30}$$

where the first integral is equal to zero because of the symmetry of the cell and of $R(\theta)$. Besides, since the origin of the polar system of coordinates O, *Figure 2*, coincides with the source point ξ at the centre of the internal cell, the second integral in (30) is also equal to zero, therefore resulting that I_{2_s} is identically equal to zero.

It is worth noting that should higher order internal cells be used the problem of correctly evaluating the singular integrals in (17) and (18) has to be tackled by using a different approach[23].

NUMERICAL EXAMPLES

To assess the performance of the proposed formulation several tests were carried out and their results compared with analytical solutions, when available. In what follows, two of these tests are described.

In Example 1 the results computed by using the present formulation are compared with an analytical solution for the problem and also with the results obtained by a transformation of coordinates which enables the use of a conventional BEM program for Laplace's equation[7]. In this latter case the original rectangular coordinates of the problem, as shown in *Figure 3*, are transformed onto the principal axes of orthotropy to eliminate the cross term $\partial^2 T/\partial x \partial y$ in (3). Then, the transformed governing differential equation becomes:

$$k_{\eta_1}\frac{\partial^2 T}{\partial \eta_1^2} + k_{\eta_2}\frac{\partial^2 T}{\partial \eta_2^2} = 0 \tag{31}$$

where η_i stands for the principal direction i of the material and k_{η_i} is the corresponding coefficient of thermal conductivity. Thus, the assumption of a homogeneous medium implies that the material is, in fact, orthotropic but its analysis with the present formulation is carried out using a set of coordinate axes that does not coincide with the principal directions of orthotropy.

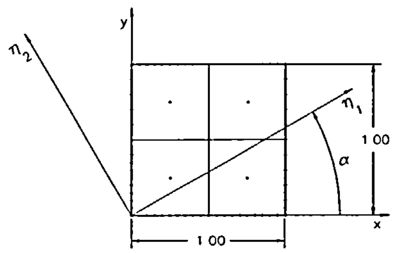

Figure 3 Discretization and principal directions of a square orthotropic domain

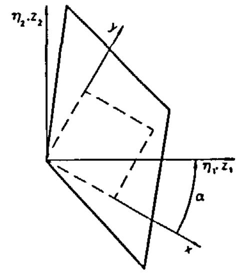
Figure 4 Transformed geometry (continuous line) due to the change of variables expressed in (32)

Following the rotation of axes, a change of variables expressed by:

$$z_i = \eta_i \sqrt{\frac{k_{\eta_i}}{k}} \tag{32}$$

is applied, where k is an arbitrary constant with the same dimensions as k_i (see Chao[24]). This change of variables makes it possible to rewrite (31) in the form of Laplace's equation, though the transformed geometry, *Figure 4*, poses the need to specify the tangencial force (i.e. $f_t = \partial T(\chi)/\partial t(\chi)$ where t is the vector tangent to the boundary at point χ) for each boundary node where Neumann boundary conditions are prescribed in the original problem. Furthermore, this transformation of coordinates can result in severe element distortion[4,25] thus making this approach not suitable for general purpose implementation[25].

The domain used as reference for the first example was a unit square with a boundary discretization of 40 constant elements of the same length and a domain discretization of four square internal cells as shown in *Figure 3*.

In the second example the number of internal cells used to model the thermal behaviour of an anisotropic strip is varied in order to verify the convergence of the method.

In both examples, according to the second principle of thermodynamics, the magnitude of the coefficients k_{ij} is limited by the requirement that[1,21]:

$$k_{xx}k_{yy} - k_{xy}^2 > 0 \tag{33}$$

Example 1

In this example a particular solution was derived from a complete second degree polynomial of the form:

$$T = a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 \tag{34}$$

which, when applied to the governing differential equation of the problem, (4), becomes:

$$T = \left[a_2(k_1 - 1) - \frac{k_2a_3}{2} \right] x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 \tag{35}$$

Taking into consideration the dimensions of the proposed model, it is possible to prescribe conditions at the boundaries according to (35). These boundary conditions can be expressed by:

$$\bar{f}_n = -\frac{\partial T}{\partial y} = -a_3x - a_5 \tag{36}$$

on the side $y = 0$;

$$\bar{T} = \left[a_2(k_1 - 1) - \frac{k_2 a_3}{2} \right] + a_2 y^2 + a_3 y + a_4 + a_5 y + a_6 \tag{37}$$

on the side $x = 1$;

$$\bar{f}_n = \frac{\partial T}{\partial y} = 2a_2 + a_3 x + a_5 \tag{38}$$

on the side $y = 1$; and:

$$\bar{T} = a_2 y^2 + a_5 y + a_6 \tag{39}$$

on the side $x = 0$.

Next, values for the coefficients of thermal anisotropy are assumed as $k_{xx} = 0.50$, $k_{xy} = 0.25$ and $k_{yy} = 0.40$. In addition, the coefficients of the polynomial are also specified as $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, $a_5 = 5$ and $a_6 = 6$. These coefficients are then applied to the expressions for determining the boundary conditions at all element nodes. The results obtained by means of the proposed formulation are presented in *Figure 5*, along with the results obtained by using transformation of coordinates and the analytical solution to the problem. It can be seen that temperature and normal force distributions are virtually coincident for all the methods. The slightly larger

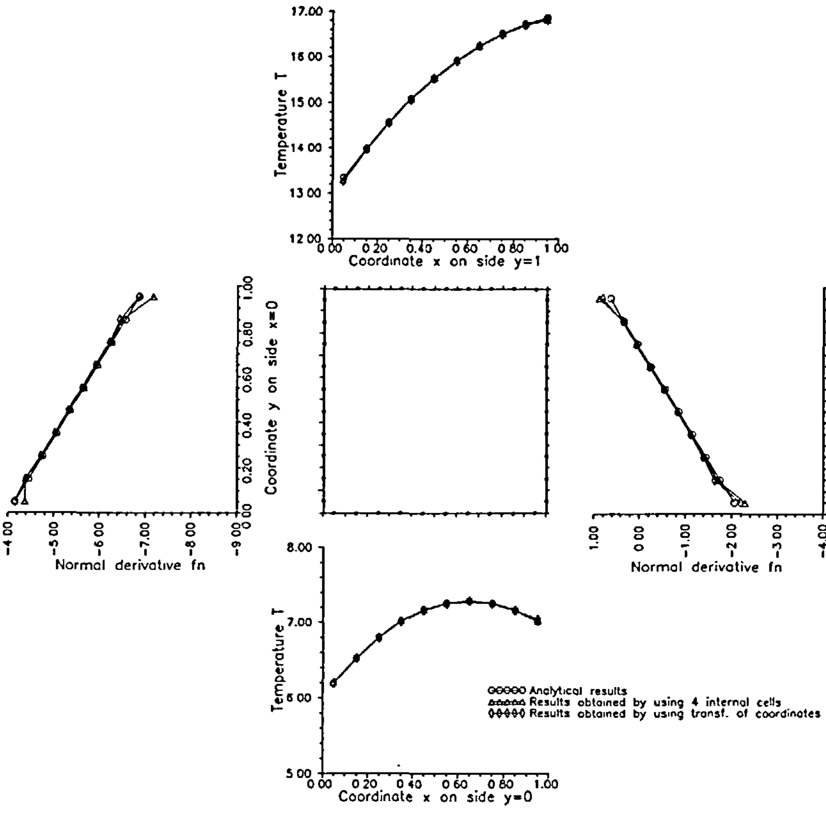

Figure 5 Results obtained for example 1

differences in the normal forces near the corners can be attributed to the use of constant elements in the BEM formulation[4].

A particularity of this problem is that the second derivatives $\partial^2 T/\partial y^2$ and $\partial^2 T/\partial x \partial y$ are constant all over the domain; thus the function $b(T)$ is correctly represented by constant cells, that are only necessary for integration purposes.

Example 2

In this example, the results obtained by means of the formulation proposed in this article are compared with the analytical solution for a steady state heat flow in an infinite homogeneous anisotropic strip obtained by Xiangzhou[20].

The strip under consideration, depicted in *Figure 6*, is replaced for modelling purposes by a strip of finite but very large length, i.e. a narrow rectangle with length equal to ten times its height. In addition, the coordinate system (x, y) conforms with the strip geometry but does not coincide with the principal directions of orthotropy of the material.

The conductivity coefficients are defined as $k_{xx} = 0.5$, $k_{xy} = 0.2236$ and $k_{yy} = 0.4$ whereas the boundary conditions are prescribed in the form of temperatures that are symmetric with respect to the y axis. These conditions are defined as:

$$T = 0, \quad y = 0 \tag{40}$$

$$T = \cos\left(\frac{\pi x}{2}\right), \quad |x| < 1, \quad y = 1 \tag{41a}$$

$$T = 0, \quad |x| > 1, \quad y = 1 \tag{41b}$$

and

$$T = 0, \quad |x| = 5 \tag{42}$$

For the numerical solution of this problem, the boundary discretization presented in *Figure 7a* is kept unchanged while three different domain meshes, shown in *Figures 7b, c* and *d* are tested in order to check the convergence behaviour of the method.

The results for the normal forces f_n obtained by using each mesh are presented along with the analytical solution for the top and bottom sides of the strip in *Figures 8* and *9*, respectively. It is of interest to note that, in *Figure 8*, the two very sharp cusps of the exact solution at $x = 1.00$ and $x = -1.00$, which are due to the discontinuity at these points of the temperature gradient in the x direction, are also obtained by means of the proposed formulation, even when very simple internal discretizations are used. In *Figure 9*, the translated symmetry owing to the effect of the anisotropy of the conductive material is satisfactorily reproduced. Moreover, it can be seen from both Figures that the overall results are in complete agreement with the analytical solution, therefore demonstrating the validity of the formulation presented.

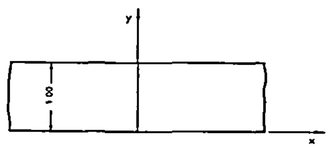

Figure 6 Anisotropic strip

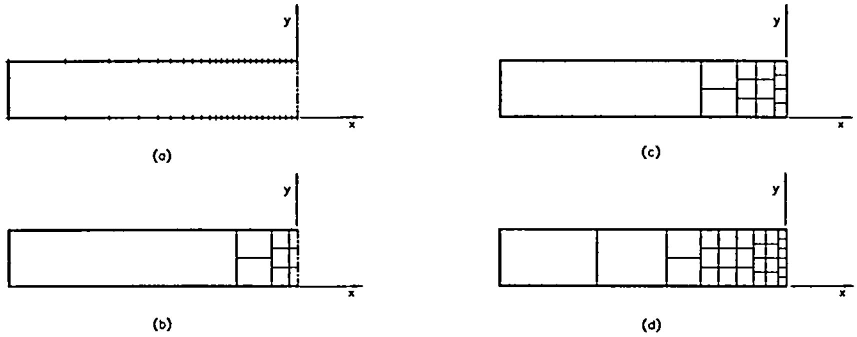

Figure 7 Boundary discretization (a); and internal meshes with 15 (b), 26 (c) and 54 (d) internal cells

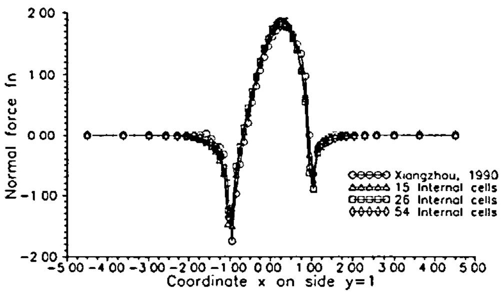

Figure 8 Normal force f_n on the boundary $y = 1$

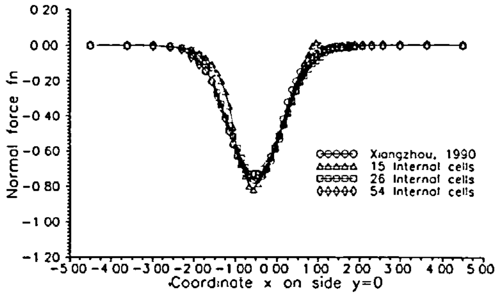

Figure 9 Normal force f_n on the boundary $y = 0$

CONCLUDING REMARKS

In this paper a system of singular integral equations for modelling heat conduction problems in homogeneous anisotropic media is formulated and solved numerically. The proposed method uses the isotropic fundamental solution for the Laplace's equation in a direct BEM approach. The strict importance of the convected terms[16,22] in the differentiation of integrals that involve singular kernels is emphasized. Some examples are presented to assess the accuracy of the proposed formulation.

The method proposed can be readily extended to three-dimensional analysis. Moreover, it can be extended to model non-homogeneous anisotropic problems by dividing the continuum into a number of subregions[5–7] within which the medium properties could be regarded as homogeneous. Bodies constituted of different orthotropic materials can also be treated by using the same approach whereas curved geometries can be represented by using higher order boundary elements and parametric internal cells provided that due care is taken in the evaluation of the Cauchy principal value integrals that arise in the proposed formulation[23].

In fact, the generality of the formulation presented here makes it possible to extend it to anisotropic elasticity, thus avoiding the complexity of its fundamental solution. Similarly, it could also be extended to elastoplasticity by including the treatment for anisotropy in the formulation recently proposed by Carrer and Telles[26]. This last formulation models transient dynamic elastic and elastoplastic behaviour of structures using the elastostatic fundamental solution. In this prospective extension, the anisotropic effects can be directly incorporated by using the internal discretization originally devised to take into account the inertial domain integral.

ACKNOWLEDGEMENTS

The authors are thankful to Prof. J. C. F. Telles, from COPPE/UFRJ, and to Prof. H. Power, from Universidad Central de Venezuela, for their communications on improper integrals. The first author would also like to acknowledge the financial support received from the Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq, of the ministry for science and technology of Brazil and to express his gratitude for the leave granted him by the Departmento de Engenharia Mecânica of the Universidade Federal de Uberlândia, which, together, have made this work possible.

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